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# DIFFERENTIAL EQUATIONS OF THE EQUILIBRIUM STATES OF THIN ELASTIC SHELLS IN THE POSTCRITICAL STAGE

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The purpose of this article is to review the basic differential equations, even if only in an approximate form of sufficient accuracy, which describe the equilibrium states of thin elastic shells after the loss of the stability of the fundamental form of equilibrium. Incidentally, there is developed a system of differential equations for the determination of the load for which there is a loss of stability of the basic form of equilibrium, and a qualitative picture of the strain after the loss of stability is investigated. However, the results obtained in the first three sections are more general and are applicable to the investigation of other problems of the equilibrium of thin elastic shells with finite displacements.

The following assumptions are made in the paper: (1) apart from finite displacements of the points of a shell in the postcritical stage, the deformations of the shell remain small; (2) the Kirchhof-Lyav kinetic hypothesis on the deformation of a shell is assumed; (3) the stressed state of the fundamental form of equilibrium is taken as without moments.

1. The Deformation of the Mean Surface of a Shell During Finite Displacements. The mean surface of a shell is considered to be related to internal coordinates  $x^1$  and  $x^2$ . The following abbreviations will be used in the paper:

$r$  - radius vector of the point  $(x^1, x^2)$ ,  $r = r(x^1, x^2)$ ;  
 $r_i$  - coordinate vectors,  $r_i = \partial r / \partial x^i$ ;  
 $a_{ij}$  - components of the basic metric form,  $a_{ij} = r_i \cdot r_j$ ; they are the components of a vector  
 $a$  - discriminant of the basic metric form,  $a = a_{11}a_{22} - a_{12}a_{21}$ ;  
 $c_{ij}$  - components of the discriminant tensor,  $c_{ii} = 0$ ,  $c_{12} = -c_{21} = \sqrt{a}$ ;  
 $n$  - unit vector along the normal,  $2n = c^{\alpha\beta} r_\alpha \times r_\beta$ ;  
 $b_{ij}$  - components of the second metric form,  $b_{ij} = n \cdot \partial^2 r / \partial x^i \partial x^j$ ;  
 $\nabla_j$  - symbol for covariant differentiation.

Let  $r^*$  be the radius vector of a point after deformation, the position of which before deformation is given by the radius vector  $r$ . In the following we shall in general mark with an asterisk all quantities

related to the deformed mean surface. The internal coordinates  $x^1$  and  $x^2$  of the point considered do not change during deformation.

The transformation of the linear elements  $dr$  of the surface during deformation into  $dr^*$  can be represented by a homogeneous affine transformation in an infinitely small neighborhood

$$dr^* = dr + dr \cdot V \quad (1.1)$$

where  $V$  is the affine operator of the transformation. The transformation of the line elements considered can be resolved in the neighborhood of an arbitrary point into a rotation without deformation and a subsequent symmetric transformation [1], caused by a pure strain, with respect to the fundamental basis  $r_i, n$ ; after the rotation we shall refer to  $p_i, m$ .

Let the line element  $dr$  be transformed during rotation into a line element  $dp$ . Using an affine operator of rotation  $P$  this can be represented as an isometric transformation in an infinitely small region

$$dp = dr + dr \cdot p \quad (1.2)$$

as a particular case of (1.2) we get

$$p_i = r_i + r_i \cdot p, \quad 2m = c^{\alpha\beta} p_\alpha \times p_\beta \quad (1.3)$$

According to the conditions of an isometric transformation

$$p_i \cdot p_j = r_i \cdot r_j \quad (1.4)$$

The pure strain is given by a symmetric affine transformation in an infinitely small region after rotation:

$$dr^* = dp + dp \cdot D \quad (1.5)$$

The affine operator  $D$  we shall call the first strain tensor of the mean surface; it is symmetric by definition, that is, designating the covariant components of  $D$  by  $\epsilon_{ij}$ , we have  $\epsilon_{ij} = \epsilon_{ji}$ .

From (1.3) it follows that the vectors  $p_i$  form a covariant tensor of the first rank with vector components, therefore  $\nabla_i p_i$  are the components of a tensor of the second rank with vector components [2].

From the resolution

$$\nabla_j p_i = (b_{ij} - \mu_{ij})_m + \zeta_{ij}^{\alpha\beta} p_\alpha \quad (1.6)$$

it follows that  $\mu_{ij}$  and  $\zeta_{ijk}$  are the components of tensors of second and third rank. We shall call the tensor  $\mu_{ij}$  the second strain tensor.

The relation of the tensor  $\zeta_{ijk}$  to the first strain tensor  $\epsilon_{sr}$  can be found from the condition of integrability

$$c^{\alpha\beta} \nabla_\beta r_i^* = 0 \quad (1.7)$$

where  $r_i^*$  is supposed to be of the form

$$r_i^* = (a_i^\alpha + \epsilon_i^\alpha) p_\alpha \quad (1.8)$$

which represents a particular case of the expression (1.5).

The vector equation (1.7) gives three consistency equations satisfied by the strain

$$c^{\alpha\beta} \nabla_\beta \epsilon_{i\alpha} + c^{\alpha\beta} (a_\alpha^\gamma + \epsilon_\alpha^\gamma) \zeta_{\gamma\beta i} = 0 \quad (1.9)$$

$$c^{\alpha\beta} \mu_{\alpha\beta} - c^{\alpha\beta} \epsilon_\alpha^\gamma (b_{\gamma\beta} - \mu_{\gamma\beta}) = 0 \quad (1.10)$$

Furthermore, by covariant differentiation with respect to  $x^k$  of equation (1.4) it is not hard to find a skew-symmetric tensor  $\zeta_{ijk}$  relative to the indices  $k, k$ :

$$\zeta_{ijk} = -\zeta_{kji}$$

since the tensor  $\zeta_{ijk}$  is defined by only two quantities, let us say  $\zeta_j$ , let

$$\zeta_{ijk} = -c_{ik} \zeta_j \quad (1.11)$$

In the case of small strain it is easy to find  $\zeta_j$  from equation (1.9):

$$\zeta_j = c^{\alpha\beta} \nabla_\beta \epsilon_{\alpha j} \quad (1.12)$$

The resolution of (1.6) must obviously satisfy the Ricci identity

$$2c^{\alpha\beta} \nabla_\beta \nabla_\alpha p_i = c^{\alpha\beta} R_{\beta\alpha i\lambda} p^\lambda \quad (1.13)$$

where  $R_{jkmn}$  are the components of the Riemann tensor of curvature. In order to develop this vector equation we note that covariant differentiation of the equations

$$p_i \cdot m = 0, \quad m \cdot m = 1$$

with respect to  $x^j$  gives us

$$\nabla_j m = - (b_{aj} - \mu_{aj}) p^a \quad (1.14)$$

The Ricci identity (1.13) gives us three scalar equations satisfied by the strain

$$c^j \gamma_c a^\beta \nabla_\beta \mu_{\gamma a} - c^a \beta_c \gamma^\rho (b^j_\rho - \mu^j_\rho) \nabla_\rho \xi_{\alpha \gamma} = 0 \quad (1.15)$$

$$c^a \beta_c \gamma^\rho (2b_{\gamma a} \mu_{\alpha \beta} - \mu_{\gamma a} \mu_{\alpha \beta} - 2 \nabla_\beta \nabla_\rho \xi_{\alpha \gamma}) = 0 \quad (1.16)$$

and to these equations may be added the consistency equation (1.10) satisfied by the strain.

In the case of small displacements one may neglect in these equations terms that are nonlinear with respect to the components of the first and second strain tensors. The equations obtained in this way for the strain agree with the equations published by A. L. Gol'denveyzer<sup>3</sup>.

2. Static Relations During Finite Displacements. In the absence of a linear theory of shells we shall investigate the equilibria of elements of the shell in the deformed, i.e., the final state.

Let  $T^i \sqrt{a}$  be the reduced principal vector of forces and  $M^i \sqrt{a}$  the reduced principal moment of forces (relative to a point on the mean surface) acting on the coordinate surface  $x^i = \text{const.}$

Starting from the condition of equilibrium of an elementary triangle cut from the shell it is not difficult to show that  $T^i$  and  $M^i$  are tensors of the first rank with vector components.

We resolve  $T^i$  and  $M^i$  along the vectors  $p_i$  and  $m$ :

$$T_i = T_i^a p_a + N^i m, \quad M_i = c^a_\beta M^i_\beta p_a \quad (2.1)$$

Since  $p_j$  is a covariant tensor of the first rank, it follows from (2.1) that  $T^{ij}$  and  $M^{ij}$  are contravariant tensors of the second rank, say tensors of the tangential forces and moments, and  $N^i$  is a contravariant vector of the transverse forces.

We consider an element of the shell bounded by the coordinate surfaces  $x^i, x^i + dx^i = \text{const.}$  and the external surface  $z = +1/2 t$ .

We designate the principal vector of all the external forces acting on this element of the shell by  $X \sqrt{a} dx^1 dx^2$  and let

$$X = X^\alpha p_\alpha + X_m \quad (2.2)$$

Furthermore, we designate the principal moment of all the external forces acting on the element of the shell considered by  $M \sqrt{a} dx^1 dx^2$  and let

$$M = M^\alpha p_\alpha \quad (2.3)$$

The conditions of equilibrium of the shell require that

$$\frac{1}{\sqrt{a}} \frac{d}{dx^\alpha} (T^\alpha \sqrt{a}) + X = \nabla_\alpha T^\alpha + X = 0 \quad (2.4)$$

$$\frac{1}{\sqrt{a}} \frac{d}{dx^\alpha} (M^\alpha \sqrt{a}) + r_\alpha^* \times T^\alpha + M = \nabla_\alpha M^\alpha + r_\alpha^* \times T^\alpha + M = 0 \quad (2.5)$$

In the developed form we obtain the equation

$$\nabla_\alpha T^{\alpha i} + c_{\beta}^i T^{\alpha\beta} c^{\gamma\rho} \nabla_\rho \varepsilon_{\gamma\alpha} - N^\alpha (b_{\alpha}^i - \mu_{\alpha}^i) + X^i = 0 \quad (2.6)$$

$$T^{\alpha\beta} (b_{\beta\alpha} - \mu_{\beta\alpha}) + \nabla_\alpha N^\alpha + X = 0 \quad (2.7)$$

$$\nabla_\alpha M^{\alpha i} + c_{\beta}^i M^{\alpha\beta} c^{\gamma\rho} \nabla_\rho \varepsilon_{\gamma\alpha} - (a_{\alpha}^i + \varepsilon_{\alpha}^i) N^\alpha + c_{\gamma}^i M^{\gamma} = 0 \quad (2.8)$$

$$c_{\beta}^{\alpha} M^{\gamma\beta} (b_{\alpha\gamma} - \mu_{\alpha\gamma}) + c_{\beta\gamma} (a_{\alpha}^{\beta} + \varepsilon_{\alpha}^{\beta}) T^{\alpha\gamma} = 0 \quad (2.9)$$

In the following, for the initial position of the shell we shall assume the position that precedes the loss of stability of the fundamental form of the equilibrium. We suppose that the initial stress is constant and without moments. We denote the tensor of the tangential forces in this state by  $T_{(0)}^{ij}$  and the components of the principal vector

of the external forces by  $X_{(0)}^i, X_{(0)}$ . After the loss of stability of the fundamental form of the equilibrium there arises a mixed stressed state, characterized by the internal conditions

$$T^{ij} = T_{(0)}^{ij} + S^{ij}, \quad M^{ij}, \quad N^i$$

which is accompanied by a field of external forces

$$X^i = X_{(0)}^i + Y^i, \quad X = X_{(0)} + Y$$

It is obvious that in the equilibrium state

$$\nabla_{\alpha} S^{\alpha i} + c_{\beta}^i (T_{(0)}^{\alpha\beta} + S^{\alpha\beta}) c^{\gamma\rho} \nabla_{\rho} \varepsilon_{\gamma\alpha} - N^{\alpha} (b_{\alpha}^i - \mu_{\alpha}^i) + \gamma^i = 0 \quad (2.10)$$

$$- T_{(0)}^{\alpha\beta} \mu_{\beta\alpha} + S^{\alpha\beta} (b_{\beta\alpha} - \mu_{\alpha\beta}) + \nabla_{\alpha} N^{\alpha} + \gamma = 0 \quad (2.11)$$

$$\nabla_{\alpha} M^{\alpha i} + c_{\beta}^i M^{\alpha\beta} c^{\gamma\rho} \nabla_{\rho} \varepsilon_{\gamma\alpha} - N^{\alpha} (a_{\alpha}^i + \varepsilon_{\alpha}^i) = 0 \quad (2.12)$$

$$c_{\beta}^{\alpha} M^{\gamma\beta} (b_{\alpha\gamma} - \mu_{\alpha\gamma}) + c_{\beta\gamma} \varepsilon_{\alpha}^{\beta} T_{(0)}^{\alpha\gamma} + c_{\beta\gamma} (a_{\alpha}^{\beta} + \varepsilon_{\alpha}^{\beta}) S^{\alpha\gamma} = 0 \quad (2.13)$$

3. The Laws of Elasticity. Taking the distribution of strain through the thickness of the shell in accordance with the hypothesis of Kirchhoff-Lyav, it is not hard to show that within the limits of accuracy of this hypothesis [4] the relations between the tensors of the internal forces and the strain tensors are not at all different from those in the theory of small displacements.

For this purpose we must assume that apart from the final displacement of the mean surface the deformations remain small, and consequently terms of the sort  $\varepsilon_{ia} \varepsilon_j^{\alpha}$ ,  $\mu_{ia} \varepsilon_j^{\alpha}$  can be neglected in comparison with the terms  $\varepsilon_{ij}$ ,  $\mu_{ij}$ . The refinement of the relations given below by non-linear terms of the type  $\mu_{ia} \mu_j^{\alpha}$  needs to be done only in the case of infinitely small strains of first order, the concept of which is given below in Section 6. However, for the shells used in engineering construction this view of strain, obviously, can only be taken in very exceptional cases, and therefore this refinement of the relations is not carried out here. The student can find it in the work of Chien [8].

Omitting the calculations, we arrive at the final form [3]

$$S_{ij} = B E_{ij}^{\alpha\beta} \varepsilon_{\alpha\beta}, \quad M_{ij} = D E_{ij}^{\alpha\beta} \mu_{\alpha\beta} \quad (3.1)$$

where

$$B = \frac{Et}{1-\nu^2}, \quad D = \frac{Et^3}{12(1-\nu^2)}, \quad E_{ijmn} = a_{im} a_{jn} + \gamma c_{im} c_{jn} \quad (3.2)$$

here E is Young's modulus,  $\nu$  is the Poisson coefficient, and t is the thickness of the shell. It is also easy to find the reverse relationships

$$\varepsilon_{ij} = B^{-1} P_{ij\alpha\beta} S^{\alpha\beta}, \quad \mu_{ij} = D^{-1} P_{ij\alpha\beta} M^{\alpha\beta} \quad (3.3)$$

where

$$B^i = \frac{1}{Et}, \quad D^i = \frac{12}{Et^3}, \quad P_{ijmn} = a_{im} a_{jn} - \gamma c_{im} c_{jn} \quad (3.4)$$

4. Determining the Critical Load. In the equilibrium state infinitely close to the fundamental equilibrium form, but qualitatively differing from it through those same external conditions,  $S^{ij}$ ,  $M^{ij}$ ,  $N^{ij}$ ,  $\epsilon^{ij}$ , and  $\mu^{ij}$  will be infinitely small quantities and therefore their products in the differential equations can be neglected. Then the equilibrium equations (2.10) - (2.13) have the form

$$\nabla_\alpha S^{a i} + c^i_{\beta} T_{(0)}^{a \beta} c^{\beta \gamma} \nabla_\gamma \epsilon_{\rho a} - b^i_a \nabla_\beta M^{\beta a} = 0 \quad (4.1)$$

$$S^{a \beta} b_{\beta a} - T_{(0)}^{a \beta} \mu_{\beta a} + \nabla_\alpha \nabla_\beta M^{\beta a} = 0 \quad (4.2)$$

$$c_{\beta a} M^{\gamma \beta} b^a_\gamma + c_{\beta \gamma} \epsilon_a^{\beta} T_{(0)}^{a \gamma} + c_{a \beta} S^{a \beta} = 0 \quad (4.3)$$

The equations for the consistency of the strain will not be different from those of the linear theory of shells; from (1.15), (1.16) and (1.10) we get

$$c^i_\gamma c^{a \beta} \nabla_\beta \mu_{\gamma a} - c^{a \beta} c^{\gamma \rho} b^i_\beta \nabla_\rho \epsilon_{a \gamma} = 0 \quad (4.4)$$

$$c^{a \beta} c^{\gamma \rho} (b_{\gamma a} \mu_{\rho \beta} - \nabla_\beta \nabla_\rho \epsilon_{a \gamma}) = 0 \quad (4.5)$$

$$c^{a \beta} \mu_{a \beta} - c^{a \beta} \epsilon_a^\gamma b_{\gamma \beta} = 0 \quad (4.6)$$

The physical relations (3.1) and (3.3) are unchanged.

The nonzero solution of the system (4.1)-(4.6) is possible only for definite values of the tensor of the tangential forces  $T_{(0)}^{ij}$ . The stressed state corresponding to the tensor  $T_{(0)}^{ij}$  will be taken as the initial stressed state. The metric tensors  $a_{ij}$  and  $b_{ij}$  are also referred to this state.

The value of the external loading for which different forms of equilibrium are possible will as usual be called the critical loading.

Certain static properties of shells are defined by the values of three geometrical parameters: the thickness of the shell  $t$ , the least radius of curvature  $R$ , and the smallest dimension of the shell  $L$ .

The shell is called thin if the ratio  $t/L = \lambda$  is small. The camber of a shell is characterized by the ratio  $L/R$ ; let

$$L/R = \lambda^r$$



It is convenient to consider shells as cambered for which  $r > 0$ ; in addition if  $r \gg 1$  we consider the shell to be extremely cambered. Cambered and extremely cambered shells will be the object of further investigation.

We take the quantity  $L$  as unity and relate the mean surface of the shell to a coordinate system having the property that the principal terms of the tensor  $a_{ij} \sim 1$ . Such an auxiliary limitation is feasible for cambered shells: it is useful for the qualitative analysis of the fundamental equations presented at the beginning of this section. For such a system of coordinates the principal value of the tensor  $b_{ij} \sim \lambda^r$ . In the following we shall say that the tensor  $t_{ij}$  is commensurable with  $\lambda^t$  if the principal term of the tensor  $t_{ij} \sim \lambda^t$ .

In the equations (4.1)-(4.6) let the principal term of the tensor  $\mu_{ij}$  have the value  $\mu$  and let the tensors  $\epsilon_{ij}$ ,  $T_{(0)}^{ij}$  be commensurable with  $\mu \lambda^e$  and  $E \lambda^{c+1}$  respectively where  $e$  and  $c$  are still unknown. Then according to (3.1) the tensors  $S_{ij}$  and  $M^{ij}$  are commensurable with  $E \mu \lambda^{e+1}$  and  $E \mu \lambda^3$  respectively. Furthermore, during covariant differentiation let the order of the unknowns be changed by a factor of  $\lambda^{-k}$ , so that, for instance,

$$\nabla_m \mu_{ij} \sim \mu \lambda^{-k}, \quad \nabla_m \epsilon_{ij} \sim \mu \lambda^{e-k}$$

Here  $k$  is also a still undefined quantity. It is not difficult to find the physical value of  $k$ : the wave length of a protruding wall of a shell will be  $\lambda \sim \lambda_{KL}$ .

We shall distinguish two types of strain during the loss of stability of the fundamental equilibrium form. Let us say that the mean surface of the shell during the loss of stability is: (a) rigid, if the tensors  $\gamma_\alpha \mu_{\rho\beta}$ , and  $\nabla_i \nabla_j \epsilon_{mn}$  are commensurable, and (b) nonrigid to first order if the tensor  $\nabla_i \nabla_j \epsilon_{mn}$  is commensurable with the tensor  $\lambda^b \gamma_\alpha \mu_{\rho\beta}$ .

In case (a) obviously

$$r = e - 2k$$

and in case (b)

$$r + 1 = e - 2k$$

### Cambered Shells ( $r \sim 0$ )

Strain of Type (a). The qualitative analysis of equation (4.2) shows that the critical loading will be smallest when  $e = 2k = 1$ ; in this case  $c$  assumes the value 1. After determining these quantities it is apparent that in equations (4.1)-(4.6) one may neglect certain nonessential terms. Namely, in the determination of the critical loading it is sufficient to consider the equations

$$\nabla_{\alpha} S^{\alpha i} = 0, \quad c_{\alpha\beta} S^{\alpha\beta} = 0 \quad (4.7)$$

$$S^{\alpha\beta} b_{\beta\alpha} - T_{(0)}^{\alpha\beta} \mu_{\beta\alpha} + \nabla_{\alpha} \nabla_{\beta} M^{\beta\alpha} = 0 \quad (4.8)$$

as the equations of equilibrium, and the equations

$$c^{\alpha\beta} \nabla_{\beta} \mu_{\alpha i} = 0, \quad c^{\alpha\beta} \mu_{\alpha\beta} = 0 \quad (4.9)$$

$$c^{\alpha\beta} c^{\gamma\rho} (b_{\gamma\alpha} \mu_{\alpha\beta} - \nabla_{\beta} \nabla_{\rho} \epsilon_{\gamma\alpha}) = 0 \quad (4.10)$$

as the equations of consistency of strain; the terms omitted are of order  $\lambda$  with respect to the terms retained<sup>1</sup>. [<sup>1</sup>Note: In the following we shall say simply that the "terms omitted are of order  $\lambda$ " if they are of this order with respect to the terms retained.] In addition to these equations we have the physical relations (3.1).

Strain of Type (b). The analysis of equation (4.2) shows that  $c$  will have a maximum for  $e = c = 1.5$ , and here  $2k = 0.5$ . It is easy to verify that in equations (4.7) the terms omitted are of order  $\lambda^{0.5}$  while in equations (4.9) of order  $\lambda^{1.5}$ . Since equation (4.5) can be written in the form

$$c^{\alpha\beta} c^{\gamma\rho} b_{\gamma\alpha} \mu_{\rho\beta} = 0$$

in order to determine  $\mu_{ij}$  we obtain the full system of linear homogeneous equations. It is the fundamental system of equations of infinitely small first-order deflections.

In Section 6 it is proved that in all cases of the loss of stability  $\mu_{ij}$  can be expressed by means of a function of the displacements  $W$ , namely

$$\mu_{ij} = \nabla_i \nabla_j W \quad (4.11)$$

An infinitely small first-order deflection with a loss of stability is realizable if  $W$  is the solution of the boundary problem that satisfies the differential equation

$$c^{\alpha\beta} c^{\gamma\rho} b_{\gamma\alpha} \nabla_{\rho} \nabla_{\beta} W = 0 \quad (4.12)$$

and certain boundary conditions imposed in the problem of the loss of stability of the fundamental equilibrium form. This solution may also be lacking, but nonetheless it is possible that in a rather large region strains of type (b) simply do not occur. Then, of course,  $1.0 < c < 1.5$ . The analysis as carried out for  $c$  and  $k$  refers to an infinitely small region of the shell, but the results obtained hold in general for all shells if the strain is qualitatively homogeneous over the whole mean surface. In the opposite case the critical loading increases on account of the more rigid regions.

#### Extremely Cambered Shell ( $r \gg 1$ )

Strain of Type (a). The qualitative analysis shows that strains of the shell are characterized by values of  $k = 0$ ,  $e = 1$ , and the parameter that defines the critical loading  $c = 2$ . The critical state is determined by equations (4.7)-(4.10), and the terms omitted are now of order  $\lambda^2$ . When there is a loss of stability an infinitely small deflection of the mean surface occurs.

Strain of Type (b). The critical loading can be determined from the equation

$$-T_{(0)} c^{\alpha\beta} \mu_{\beta\alpha} + \nabla_{\alpha} \nabla_{\beta} M^{\alpha\beta} = 0 \quad (4.13)$$

Here the term omitted is of order  $\lambda$ . It is easily found that in the present case also,  $c = 2$ ,  $k = 0$ .

These results were first obtained by Zoelly [10] who applied an energetic method to determine the critical loading. We note that for the parameter  $\lambda$  in Zoelly's work the ration  $t/R$  is taken.

Conclusions. The preceding analysis shows that the critical loading is always determined by equations (4.7)-(4.10) to an accuracy of at least  $\lambda^{0.5}$ . In fact for a loss of stability in shells, we cannot determine in advance for any particular practical case that the strain will be of type (b), and therefore it is proper in this case to start from the more general equations (4.7)-(4.10).

At the end of the paper equations (4.7)-(4.10) will be presented in a form more convenient for integration.

5. Equilibrium of a Shell in the Postcritical Stage. The equilibrium state of a shell after the loss of stability of the fundamental (without moments) equilibrium state is determined by the

equilibrium equations (2.10)-(2.13) and the strain consistency equations (1.15), (1.16) and (1.10), and also the components of the tensors of strain and internal forces are connected by the physical relations (3.1) or (3.3). However this system is rather complex for carrying out computations, and we shall attempt through qualitative analysis of the postcritical stage to retain only essential terms in these equations.

We consider a certain equilibrium state of a shell in the postcritical stage. In this state let the tensors  $\epsilon_{ij}$  and  $\mu_{ij}$  be commensurable with  $\lambda^e$  and  $\lambda^m$  respectively, then according to (3.1) the tensors  $S_{ij}$  and  $M_{ij}$  are commensurable with  $E\lambda^{e+1}$  and  $E\lambda^{m+3}$  respectively. The quantities  $e$  and  $m$  are bounded below, since all stresses must be within the elastic limits. For the sake of simplification we take  $m > 0$  and  $e > 1$ ; these values hold if  $\sigma\rho/E = \lambda$  where  $\sigma\rho$  is the limit of the proportionality. First of all it is obvious that the equilibrium equations (2.10)-(2.13) can be replaced by the system

$$\nabla_\alpha S^{\alpha i} - (b^i_\alpha - \mu^i_\alpha) \nabla^\beta M^{\beta\alpha} + Y^i = 0 \quad (5.1)$$

$$S^{\alpha\beta} b_{\beta\alpha} - (T_{(0)}^{\alpha\beta} + S^{\alpha\beta}) \mu_{\beta\alpha} + \nabla_\alpha \nabla_\beta M^{\beta\alpha} + Y = 0 \quad (5.2)$$

$$c^\alpha_\beta M^{\alpha\beta} (b_{\alpha\gamma} - \mu_{\alpha\gamma}) + c_{\alpha\beta} S^{\alpha\beta} = 0 \quad (5.3)$$

but further simplifications are as yet unfounded.

In order for our discussion to be applicable to the widest area, we shall assume that the external loading does not change essentially after a loss of stability.

Immediately after a loss of stability, as we saw in Section 4, the strain of a shell is characterized by the fact that covariant differentiation increases the order of the unknown quantities, generally speaking by a factor of  $\lambda^{-k}$ . It should not be assumed however with an increase of the deformation in the postcritical stage this property would change numerically even if only by a little bit, since this would require a qualitative change of the field of the displacements. The load for which such a qualitative change of the field of displacements occurs is called the second critical loading. We shall consider the equilibrium of a shell in a stage between the first two critical loadings.

The classification of the states of shells are based on the properties of the consistency equations (1.16). We shall say that strains of the mean surface of the shell with relatively large

displacements, the field of which is qualitatively determined with the loss of stability of the fundamental equilibrium form is (A) rigid if the tensors  $b_{ij} \mu_{km}$  or  $\mu_{ij} \mu_{km}$  and  $\nabla_i \nabla_j \epsilon_{km}$  are commensurable, and (B) nonrigid of first order if the tensor  $b_{ij} \mu_{km}$  or  $\mu_{ij} \mu_{km}$  is commensurable with the tensor  $\lambda^{-1} \nabla_i \nabla_j \epsilon_{km}$ . Consequently for strains of type (A)

$$r + m \sim e - 2k \text{ if } r < m$$

$$2m \sim e - 2k \text{ if } r > m$$

for strains of type (B)

$$r + m \sim e - 2k - 1 \text{ if } r < m$$

$$2m \sim e - 2k - 1 \text{ if } r > m$$

In the following we shall consider the equilibrium state of cambered and extremely cambered shells in the postcritical stage with finite displacements with rigid (A) and nonrigid (B) strains of the mean surface, distinguished for this case from the rigid (a) and nonrigid (b) strains of the mean surface for the loss of stability.

However, a consideration of the change from a nonrigid strain of type (b) into a rigid strain of type (A) or on the other hand from (a) to (B) requires an analysis of the strain in the transition region, which we shall call the semirigid strain of the mean surface and denote by (C).

Quantitatively the strain of type (C) is determined by the relations

$$r + m \sim e - 2k - 0.5 \text{ if } r < m$$

$$2m \sim e - 2k - 0.5 \text{ if } r > m$$

The analysis carried out concerning the order of the individual quantities in the basic equation for finite displacements shows that in all cases it is possible to employ the simplified equations

$${}^c \alpha \beta \nabla_\beta \mu_{i\alpha} = 0, \quad {}^c \alpha \beta \mu_{\alpha\beta} = 0 \quad (5.4)$$

$$\nabla_\alpha S^{\alpha i} + Y^i = 0, \quad {}^c \alpha \beta S^{\alpha\beta} = 0 \quad (5.5)$$

instead of (1.15), (1.10) and (5.1), (5.3). In Table 1 are shown the orders of the terms omitted with respect to those retained; if, for instance, they are of order  $\lambda^{0.5}$  this is shown in the table by the number 0.5.

Occasionally certain terms in equations (1.16) and (5.2) are also unimportant. The order of these terms is also shown in the table in the same form as mentioned above.

We have limited our consideration of equilibrium states in the postcritical stage during large displacements, the field of which is qualitatively determined during a loss of stability of the fundamental equilibrium form. Thus, the wave length of a protruding wall is already defined for critical loading.

However in the postcritical stage there may exist other equilibrium forms, for which the field of displacements is qualitatively different from those considered. For these forms there might be a quite different wave length for a protruding wall, and therefore one should not consider that equations (1.16), (5.4), (5.2) and (5.5) are sufficiently complete for the definition of the equilibrium forms mentioned.

The tensor  $\epsilon_{ij}$  and consequently also the tensor  $S^{ij}$  also contain, generally speaking, parts which do not increase with differentiation, but this does not change the results of the analysis.

6. The Introduction of the Displacement and Stress Functions. A very simple qualitative investigation of the postcritical stage of both cambered and extremely cambered shells has shown that the most general system of differential equations is required to consider cases 1 and 7 of Table 1 where the basic system consists of equations (1.16), (5.4), (5.2) and (5.5) with all terms. All the remaining alternatives are obtained from this system by omitting certain nonessential terms and not adding new ones.

Practically, it is always proper to use the system (1.16), (5.4), (5.2) and (5.5) because in the solution of actual problems we cannot tell in advance whether the strain of the mean surface for large displacements will be nonrigid or not. However, we now know that for non-rigid strain, new terms in the equations are not necessary.

The nature of the strain in the postcritical region or rather a small gaussian curvature of extremely cambered shells makes it possible to express the components of the second strain tensor by means of a scalar function of the displacement  $W$ :

$$\mu_{ij} = \nabla_i \nabla_j W \quad (6.1)$$

Actually the right-hand equation of (5.4) satisfies this identically and the left-hand one does to an accuracy of  $\lambda^{2r+2k}$ .

In the same way it can be shown the equations (5.5) in case  $y^i = 0$  are satisfied to an accuracy of at least  $\lambda^{2r+2k}$  if

$$s_{ij} = c^i \alpha_{cj} \beta \nabla_\alpha \nabla_\beta F \quad (6.2)$$

$F$  is an arbitrary scalar function, the so-called stress function.

Functions of the displacement  $W$  and the stress  $F$  are determined from equations (1.16) and (5.2). Expressing  $\epsilon_{ij}$  through  $F$  and  $M^{ij}$  through  $W$  using the physical relations (3.4) and (3.1), we get two equations

$$2B^i_a \alpha \gamma_a \beta \rho \nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_\rho F - c^i \alpha \gamma_c \beta \rho (2b_{\alpha\beta} \nabla_\gamma \nabla_\rho W - \nabla_\alpha \nabla_\beta W \cdot \nabla_\gamma \nabla_\rho W) = 0 \quad (6.3)$$

$$c^i \alpha \gamma_c \beta \rho b_{\alpha\beta} \nabla_\gamma \nabla_\rho F - c^i \alpha \gamma_c \beta \rho \nabla_\gamma \nabla_\rho F \cdot \nabla_\alpha \nabla_\beta W - T_{(0)} \alpha \beta \nabla_\alpha \nabla_\beta W + D a^i \alpha \gamma_a \beta \rho \nabla_\beta \nabla_\alpha \nabla_\gamma \nabla_\rho W + Y = 0 \quad (6.4)$$

In the derivation of these equations unimportant terms are introduced on changing the order of differentiation.

Equations (6.3) and (6.4) are the fundamental differential equations for determining equilibrium states in the postcritical stage of shell if the tangential component of the external load  $y^i$  has a negligible value in the left-hand equation of (5.5).

In the particular case of a flat plate the basic equations (6.3) and (6.4) do not differ from the equations of von Karman for flat plates [5]. These equations for the case of a circular cylindrical shell were first derived by L. Donnell<sup>6</sup> and later by Tsien [7]. Finally these equations were obtained by Chien [8] as a particular case (6.12) during the investigation of thin plates and shells.

The basic equations for the determination of the critical loading (4.7)-(4.10) can be brought to the following form in the same way

$$B^i_a \alpha \gamma_a \beta \rho \nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_\rho F - c^i \alpha \gamma_c \beta \rho b_{\alpha\beta} \nabla_\gamma \nabla_\rho W = 0 \quad (6.5)$$

$$c^i \alpha \gamma_c \beta \rho b_{\alpha\beta} \nabla_\gamma \nabla_\rho F - T_{(0)} \alpha \beta \nabla_\alpha \nabla_\beta W + D a^i \alpha \gamma_a \beta \rho \nabla_\beta \nabla_\alpha \nabla_\gamma \nabla_\rho W = 0 \quad (6.6)$$

Equations (6.5) and (6.6) in an orthogonal coordinate system were first published by V. Z. Vlasov [9].

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TABLE 1

Case	For critical loading	For finite displacements	Strain State	Terms retained (marked X) and the order of terms omitted in the equations (indicated by numbers)	(1.16)	(5.2)	(1.15) or (1.10)	(5.1) or (5.3)	Order of omitted quantities in equations
1	a	A	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
2	a	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
3	a	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
4	b	A	0.5	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
5	b	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
6	b	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
Cambered Shells									
1	a	A	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
2	a	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
3	a	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
4	b	A	0.5	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
5	b	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
6	b	B	0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
Extremely Cambered Shells									
7	a,b	A	1.0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
8	a,b	A	0.5	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
9	a,b	B	1.0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
10	a,b	B	0.5	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
11	a,b	B	1.0	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			
12	a,b	B	0.5	$\begin{matrix} p_{1j} \\ \mu_{km} \end{matrix}$	$\begin{matrix} \Delta_i \\ \Delta_j \end{matrix}$	$\begin{matrix} s_{ab} \\ \mu_{ba} \\ \mu_{ba} \\ \mu_{ba} \end{matrix}$			

**\*Note:** The tensor  $S_{ij}$  drops out of the fundamental equations, therefore equations (5.1) and (5.3) are unimportant.